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A FRACTIONAL VERSION OF THE ERDŐS–FABER–LOVÁSZ CONJECTURE

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Let H be any hypergraph in which any two edges have at most one vertex in common. We prove that one can assign non-negative real weights to the matchings of H, summing to at most $|V(H)|$, such that for every edge the sum of the weights of the matchings containing it is at least 1. This is a fractional form of the Erdős–Faber–Lovász conjecture, which in effect asserts that such weights exist and can be chosen 0,1-valued. We also prove a similar fractional version of a conjecture of Larman, and a common generalization of the two.

1. Introduction

A *hypergraph* H consists of a finite set $V(H)$ of *vertices* and a set $E(H)$ of non-empty subsets of $V(H)$ called *edges*. (This definition is slightly non-standard, but convenient for us.) It is *nearly-disjoint* if $|e \cap f| \leq 1$ for every pair e, f of distinct edges. A *matching* in H is a subset of $E(H)$ of mutually disjoint edges. There is a well-known conjecture due to Erdős, Faber and Lovász [3] (for resolution of which Erdős currently offers \$500) equivalent [8] to the following.

(1.1) (Conjecture) *For every nearly-disjoint hypergraph H, $E(H)$ is the union of $|V(H)|$ matchings.*

Let $\mathbb{R}^+$ be the set of non-negative real numbers. Our first result is that

(1.2) *Let $\mathcal{M}$ be the set of all matchings of a nearly-disjoint hypergraph H. There is a function $f : \mathcal{M} \to \mathbb{R}^+$ satisfying:*
(i) $\sum_{M\in\mathcal{M}} f(M) \leq |V(H)|$, and
(ii) $\sum(f(M) : M \in \mathcal{M},\, M \ni e) \geq 1$ *for every edge* $e \in E(H)$.

It is easy to see that (1.1) is equivalent to the assertion that the function f in (1.2) may be chosen 0,1-valued. By linear programming duality, (1.2) is equivalent to

(1.3) *Let H be a nearly-disjoint hypergraph, and let $p : E(H) \to \mathbb{R}^+$ satisfy $\sum_{e\in M} p(e) \leq 1$ for every matching M. Then* $\sum_{e\in E(H)} p(e) \leq |V(H)|$.

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These results were conjectured in [8], where (1.3) was proved for constant p; that is, it was shown that some matching contains at least $|E(H)|/|V(H)|$ edges. But the proof here of (1.3) is considerably simpler, because we are able to exploit the properties of a worst case p. We can, in fact, say exactly when (1.2) is sharp but the proof is fairly involved and we just state the result:

(1.4) (1.2) *is sharp for H (in the sense that every such function f attains equality in* (1.2) (i)*) if and only if either H is a complete graph with $|V(H)|$ odd, or $|E(H)| = |V(H)|$ and every two edges of H intersect.*

Our second result is related to the following conjecture of Larman [6].

(1.5) (Conjecture) *Let $k \geq 0$ be an integer, and let H be a hypergraph such that $|e \cap f| \geq k$ for all distinct e, $f \in E(H)$. Then $E(H)$ may be partitioned into $|V(H)|$ subsets such that $|e \cap f| \geq k + 1$ for all distinct e, f belonging to the same subset.*

For $k = 0$ this is obvious, but even for $k = 1$ it remans open. Incidentally, it is tempting to propose a generalization of (1.1) parallel to (1.5), but that is false. Z. Füredi and the second author observed that if H is the Steiner system, with 22 vertices, in which every edge has 6 vertices and every 3-set lies in a unique edge, then,

(i) $|e \cap f| \leq 2$ for all distinct e, $f \in E(H)$ (indeed, $|e \cap f| = 2$ or 0)

(ii) $E(H)$ cannot be partitioned into $|V(H)| = 22$ subsets such that $|e \cap f| \leq 1$ for edges in the same subset; indeed, it cannot even be partitioned into 38 such subsets, since $|E(H)| = 77$ and among any three edges, some two have two common vertices.

A hypergraph is *intersecting* if $e \cap f \neq \emptyset$ for all e, $f \in E(H)$. It is *non-singleton* if $|e| \geq 2$ for all $e \in E(H)$. Füredi and the second author (among others) have proposed the following strengthening of (1.5) in the case $k = 1$. (Its analogue for large k is false.)

(1.6) (Conjecture) *Let H be an intersecting non-singleton hypergraph. Then there is a loopless graph G with $V(G) = V(H)$ and $|E(G)| \leq |V(H)|$ such that every edge of H contains an edge of G.*

The following fractional form of (1.6) was proved (unpublished) by Füredi and the second author. We shall prove a common generalization of this and (1.2).

(1.7) *Let H be an intersecting non-singleton hypergraph, and let K be the set of all 2-elements subsets of $V(H)$. Then there is a function $f : K \to \mathbb{R}^+$ satisfying:*

(i) $\sum_{k \in K} f(k) \leq |V(H)|$, *and*

(ii) $\sum(f(k) : k \in K,\ k \subseteq e) \geq 1$ *for every edge $e \in E(H)$.*

Incidentally, N. Alon and the second author have proved the following, which can be shown to imply (1.7).

(1.8) *Let H be an intersecting non-singleton hypergraph, and for each $v \in V(H)$ let $\mu(v)$ be the maximum, over all $u \in V(H) - \{v\}$, of the number of edges including $\{u, v\}$. Then* $\sum_{v \in V(H)} \mu(v) \geq |E(H)|$.

However, the proof of (1.8) is long and complicated, while as we shall see, (1.7) can be proved quite simply. By linear programming duality, (1.7) is equivalent to

(1.9) *Let H be an intersecting non-singleton hypergraph, and let $q : E(H) \to \mathbb{R}^+$ satisfy*

$$\sum(q(e) : e \in E(G),\ \{u,v\} \subseteq e) \leq 1$$

for all distinct u, $v \in V(H)$. Then $\sum_{e\in E(H)} q(e) \leq |V(H)|$.

We shall prove the following common generalization of (1.3) and (1.9).

(1.10) *Let H be a non-singleton hypergraph, and let p, $q : E(H) \to \mathbb{R}^+$ satisfy*

(i) $\sum_{e\in M} p(e) \leq 1$ *for every matching M*

(ii) $\sum(q(e) : e \in E(H),\ \{u,v\} \subseteq e) \leq 1$ *for all distinct u, $v \in V(H)$.*

Then $\sum_{e\in E(H)} p(e)q(e) \leq |V(H)|$.

(1.10) implies (1.9) by setting $p \equiv 1$. Moreover (1.3) may easily be reduced to its special case when H is non-singleton, which follows from (1.10) by setting $q \equiv 1$.

2. Proof of the Theorem

Our proof of (1.10) is by an application of a lemma below due to Motzkin [7]. Motzkin originally used this to give a new proof of the de Bruijn–Erdős theorem [2], which is (1.10) when H is intersecting and nearly-disjoint and $p \equiv q \equiv 1$, and our proof may be considered a generalization of Motzkin's. We prove the lemma for the reader's convenience. A graph is *simple* if it has no loops or multiple edges; $u \sim v$ denotes that vertices u, v are adjacent; and $d(v)$ denotes the number of edges incident with a vertex v.

(2.1) (Motzkin's Lemma) *Let G be a simple bipartite graph with bipartition (X, Y), such that $X \neq \emptyset$ and no vertex in X is adjacent to every vertex in Y. Then there exist $x \in X$ and $y \in Y$ such that $x \not\sim y$ and $|Y| \cdot d(y) \leq |X| \cdot d(x)$.*

Proof. If some $y \in Y$ is adjacent to every vertex in X the result follows by deletion of y, and induction. Otherwise,

$$\sum\left(\frac{|X|\cdot d(x) - |Y|\cdot d(y)}{(|X| - d(y)(|Y| - d(x))} : x \in X,\ y \in Y,\ x \not\sim y\right) =$$

$$\sum\left(\frac{|Y|}{|Y| - d(x)} - \frac{|X|}{|X| - d(y)} : x \in X,\ y \in Y,\ x \not\sim y\right) = \sum_{x\in X}|Y| - \sum_{y\in Y}|X| = 0$$

and since there exist $x \in X$ and $y \in Y$ with $x \not\sim y$ it follows that for some such pair x, y the corresponding summand of the first term is non-negative, as required. ∎

We deduce

(2.2) *Let G be a simple bipartite graph with bipartition (X, Y), such that $X \neq \emptyset$, and let $f : Y \to \mathbb{R}^+$ be such that no vertex in X is adjacent to every $y \in Y$ with $f(y) \neq 0$. Then there exist $x \in X$ and $y \in Y$ such that $x \not\sim y$, $f(y) \neq 0$, and*

$$d(y) \cdot \sum_{z\in Y} f(z) \leq |X| \cdot \sum(f(z) : z \in Y,\ z \sim x).$$

Proof. By rational approximation and clearing denominators we may assume that f is integral. Replace each $y \in Y$ by $f(y)$ copies of itself; then the result follows from (2.1). ∎

Proof of (1.10).

(If we set $q \equiv 1$ in the following, it becomes a proof of (1.3). We suggest that it should first be read in that light, since it may be easier to follow.) Fix q as in (1.10), and then choose p as in (1.10) with $\sum_{e \in E(H)} p(e)q(e)$ maximum. We may assume that $p(e) \neq 0$ for each edge e, by deleting from H all edges with $p(e) = 0$. By linear programming duality there exists $f : \mathcal{M} \to \mathbb{R}^+$ such that

(i) $\sum_{M \in \mathcal{M}} f(M) = \sum_{e \in E(H)} p(e)q(e)$

(ii) $\sum(f(M) : M \in \mathcal{M},\ M \ni e) \geq q(e)$ for every $e \in E(H)$.

By complementary slackness (since $p(e) \neq 0$ for each edge e), we deduce that equality holds in (ii) for each edge $e \in E(H)$, and

(iii) $\sum_{e \in M} p(e) = 1$ for each $M \in \mathcal{M}$ with $f(M) \neq 0$.

(Thanks to Klaus Truemper for this formulation.) We denote the union of the members of a matching M by $\cup M$. Suppose that some vertex v of H belongs to $\cup M$ for each matching M with $f(M) \neq 0$. Then

$$\begin{aligned}\sum_{e \in E(H)} p(e)q(e) &= \sum_{M \in \mathcal{M}} f(M) = \sum \left(\sum(f(M) : M \in \mathcal{M}, M \ni e) : e \in E(H), e \ni v\right)\\ &= \sum(q(e) : e \in E(H), e \ni v) \leq \sum(q(e)\cdot(|e|-1) : e \in E(H), e \ni v)\\ &= \sum\left(\sum(q(e) : e \in E(H), \{u,v\} \subseteq e) : u \in V(H) - \{v\}\right)\\ &\leq \sum_{u \in V(H)-\{v\}} 1 \leq |V(H)|\end{aligned}$$

as required. We assume then that no such vertex v exists. By applying (2.2) to the bipartite graph G with bipartition $(V(H), \mathcal{M})$, in which $v \in V(H)$ is adjacent to $M \in \mathcal{M}$ if $v \in \cup M$, we deduce that there exists $v \in V(H)$ and $N \in \mathcal{M}$ such that $v \notin \cup N$, $f(N) \neq 0$ and

$$|\cup N| \cdot \sum_{M \in \mathcal{M}} f(M) \leq |V(H)| \cdot \sum(f(M) : M \in \mathcal{M},\ \cup M \ni v).$$

But

$$\begin{aligned}\sum(f(M) : M \in \mathcal{M}, \cup M \ni v) &= \sum\left(\sum(f(M) : M \in \mathcal{M}, M \ni e) : e \in E(H), e \ni v\right)\\ &= \sum(q(e) : e \in E(H), e \ni v).\end{aligned}$$

Moreover, since $f(N) \neq 0$ it follows from (iii) that $\sum_{e \in N} p(e)$ is maximum, and so for all edges e containing v, $N \cup \{e\}$ is not a matching, that is, $e \cap \{\cup N\} \neq \emptyset$. Hence

$$\sum(q(e) : e \in E(H), e \ni v) \leq \sum\left(\sum(q(e) : e \in E(H), \{u, v\} \subseteq e) : u \in \cup N\right)$$
$$\leq \sum_{u \in \cup N} 1 = |\cup N|.$$

On combining these three observations, we obtain

$$|\cup N| \cdot \sum_{M \in \mathcal{M}} f(M) \leq |V(H)| \cdot \sum(q(e) : e \in E(H),\ e \ni v) \leq |V(H)| \cdot |\cup N|$$

and the theorem follows from (i). ∎

3. Remarks

For (simple) graphs the Erdős–Faber–Lovász conjecture (1.1) is contained in Vizing's theorem [9], the following.

(3.1) *If G is a simple graph of maximum degree D then $E(G)$ is the union of $D+1$ matchings.* ∎

One might try to extend this to hypergraphs. For a general hypergraph H, we define

$$N(v) = \cup(e : e \in E(H),\ v \in e)$$

for each $v \in V(H)$, and define $\Delta(H) = \max(|N(v)| : v \in V(H))$. If H is a simple graph of maximum degree D then $\Delta(H) = D + 1$, and so (3.1) is a special case of the following conjecture

(3.2) (Conjecture) *For every nearly-disjoint hypergraph H, $E(H)$ is the union of $\Delta(H)$ matchings.*

This has been proposed many times, for example by Meyniel (unpublished), Berge [1] and Füredi [5]. Of course, a consequence would be the corresponding generalization of (1.2), obtained from (1.2) by replacing $|V(H)|$ by $\Delta(H)$. As far as we know this conjecture first appeared in [4].

It does not seem impossible that our proof of (1.2) can be modified to give this stronger form of (1.2). The main difficulty to be overcome is that we have only been able to make use of the maximality of certain matchings, and not that they have maximum weight.

We would also like to propose a further strengthening of (1.2), the following:

(3.3) (Conjecture) *Let $\mathcal{M}$ be the set of matchings of a nearly-disjoint hypergraph H, and let $w : E(H) \to \mathbb{R}^+$ satisfy $\sum(w(e) : e \in E(H),\ e \ni v) \leq 1$ for each $v \in V(H)$. Then there exists $f : \mathcal{M} \to \mathbb{R}^+$ such that*

(i) $\sum_{M \in \mathcal{M}} f(M) \leq 1$, *and*

(ii) $\sum(f(M) : M \in \mathcal{M},\ M \ni e) \geq \frac{w(e)}{|e|-1+w(e)}$ *for each $e \in E(H)$.*

This implies the previous strengthening of (1.2) by taking $w(e) = \frac{|e|-1}{\Delta(H)-1}$. (We have proved (3.3) when H is a graph, using Edmonds' matching polytope theorem.)

One final remark. C. Berge [1] suggested trying at least to prove (3.2) for intersecting hypergraphs, that is,

(3.4) *If H is intersecting and nearly-disjoint then* $|E(H)| \le \Delta(H)$.

This was proved by Füredi [5] and independently by L. Lovász and the present authors (unpublished). We mention the problem here to point out that it too is a consequence of Motzkin's lemma.

Proof of (3.4). For $v \in V(H)$ we denote by $d(v)$ the number of edges of H containing v. Choose $v \in V(H)$ with $d(v)$ maximum, and let G be the bipartite graph with bipartition $(N(v), E(H))$, in which $u \in N(v)$ and $e \in E(H)$ are adjacent if $u \in e$. Since it is easy to see that except in trivial cases G satisfies the hypotheses of Motzkin's lemma, we only need to show that if $u \in N(v)$, $e \in E(H)$ and $u \notin e$ then $d(u) \le |e \cap N(v)|$. But this is straightforward: if $v \notin e$ then $d(u) \le d(v) = |e \cap N(v)|$, while if $v \in e$ then $d(u) \le |e| = |e \cap N(v)|$. ∎

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